

Wykład 11a

Ciekły hel czyli kondensacja
Bosego-Einsteina i elektrony w
przewodniku czyli mechanika
statystyczna fermionów

Kondensacja Bosego-Einsteina

$$\Xi = \prod_i \frac{1}{1 - e^{-\beta(\varepsilon_i - \mu)}}$$

$$\ln \Xi = - \sum_i \ln [1 - e^{-\beta(\varepsilon_i - \mu)}]$$

Dla niezbyt niskich temperatur

$$\varepsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$\ln \Xi = -s \frac{1}{\beta} \int \ln \left[1 - e^{-\frac{\beta p^2}{2m}} e^{\beta \mu} \right] \frac{d^3 p}{(2\pi\hbar)^3} V$$

s – degeneracja spinowa

$$\rho = \frac{\langle N \rangle}{V} = \frac{1}{V} \frac{\partial \ln \Xi}{\partial \mu} = s \int \frac{e^{-\frac{\beta p^2}{2m}} e^{\beta \mu}}{1 - e^{-\frac{\beta p^2}{2m}} e^{\beta \mu}} \frac{d^3 p}{(2\pi\hbar)^3}$$

$$\lambda = e^{\beta\mu}, x = \beta p^2 / 2m$$

$$\rho = s \int_0^\infty \frac{e^{-x^2} \lambda}{1 - e^{-x^2} \lambda} \left[\frac{x^2}{2\pi^2 \hbar^3} \left(\frac{2m}{\beta} \right)^{3/2} \right] dx =$$

$$s \frac{1}{4\pi^2 \hbar^3} \left(\frac{2m}{\beta} \right)^{3/2} \left[\int_{-\infty}^{+\infty} x^2 \left\{ \lambda e^{-x^2} + \lambda^2 e^{-2x^2} + \dots \right\} dx \right] =$$

$$s \left(\frac{mk_B T}{2\pi \hbar^2} \right)^{3/2} \left[\lambda + \frac{\lambda^2}{2^{3/2}} + \frac{\lambda^3}{3^{3/2}} + \dots \right] = s \left(\frac{mk_B T}{2\pi \hbar^2} \right)^{3/2} \zeta_{3/2}(\lambda)$$

$$\zeta_r(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n^r}$$

Stąd możemy
wyznaczyć λ
jako funkcję
gęstości

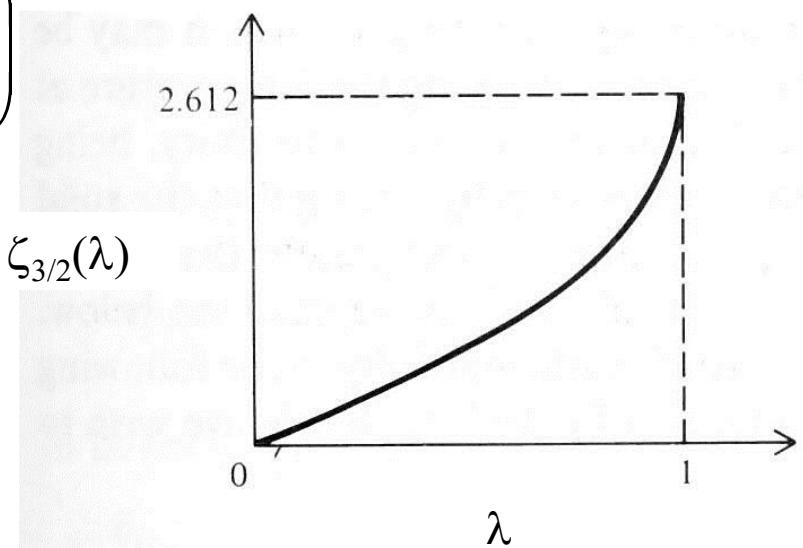
$$E = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \beta} = \sum_k \frac{\lambda \varepsilon_k e^{-\beta \varepsilon_k}}{1 - e^{-\beta \varepsilon_k}} = s \int \frac{e^{-p^2/2m} e^{\beta \mu} p^2 / 2m}{1 - e^{-\beta p^2/2m} e^{\beta \mu}} V \frac{d^3 p}{(2\pi\hbar)^3} =$$

$$s \frac{3}{2} k_B T \left(\frac{mk_B T}{2\pi\hbar^3} \right)^{3/2} V \zeta_{5/2}(\lambda) = \frac{3}{2} sk_B TV \frac{1}{\Lambda^3} \zeta_{5/2}(\lambda)$$

Dla małych wartości λ $\zeta_{3/2}(\lambda) \approx \zeta_{5/2}(\lambda) \approx \lambda$

$$\rho \approx s \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \lambda \Rightarrow \lambda \approx \frac{1}{s\rho} \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{3/2}$$

$$E \approx \frac{3}{2} k_B T \frac{V}{\rho} = \frac{3}{2} N k_B T$$



W wyrażeniu na gęstość $\zeta_{3/2}(\lambda) = \lambda + \frac{\lambda^2}{2^{3/2}} + \frac{\lambda^3}{3^{3/2}} + \dots$

jest rozbieżne dla $\lambda > 1$ ponieważ dla niskich temperatur sumowania nie można zastąpić całkowaniem.

Definiujemy temperaturę krytyczną kondensacji Bosego-Einsteina, która odpowiada $\lambda = 1$; wtedy $\zeta_{3/2}(1) = \zeta_{5/2}(1) = 2.612\dots$

$$\rho = \frac{N}{V} = s \left(\frac{mk_B T_{kr}}{2\pi\hbar^2} \right)^{3/2} \zeta_{3/2}(1)$$

$$T_{kr} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{\rho/s}{2.612\dots} \right)^{3/2}$$

Niskie temperatury (niezbyt małe λ , $T < T_{kr}$)

$$\ln \Xi = -\sum_i \ln [1 - e^{-\beta(\varepsilon_i - \mu)}]$$

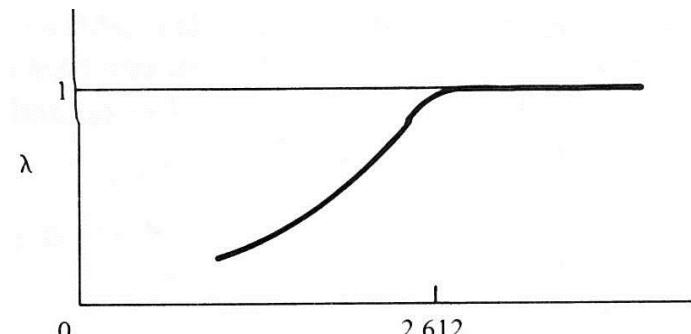
$$n_\alpha = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \varepsilon_\alpha} = \frac{1}{e^{-\beta\mu} e^{\beta\varepsilon_\alpha} - 1}$$

$$\varepsilon_\alpha - \mu > 0, \quad \varepsilon_\alpha - \mu \ll \beta, \alpha > 0$$

$$n_0 = \frac{1}{e^{-\beta\mu} e^{\beta\varepsilon_0} - 1}, \quad n_1 \ll n_0$$

Mogliśmy założyć, że $\varepsilon_0 = 0$

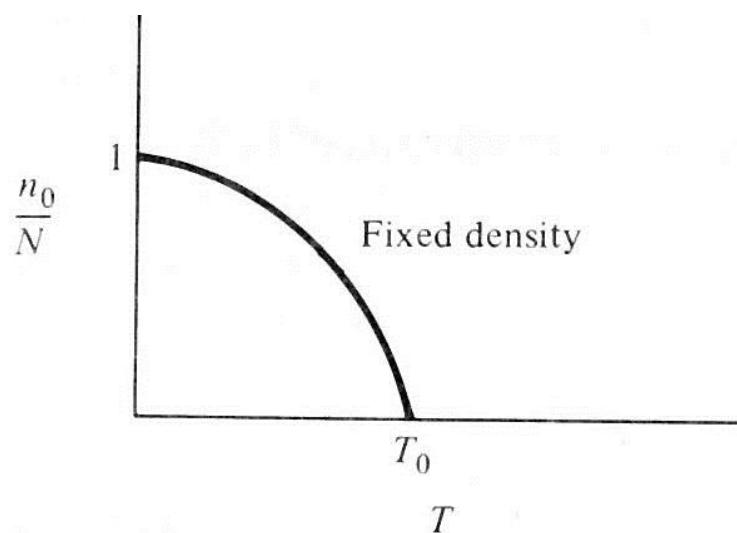
Żeby to było spełnione to μ musi być bardzo małe w porównaniu z jakąkolwiek względną ε



$$N_{wzb} = \langle N - n_0 \rangle = \sum_{i=1}^{\infty} \frac{1}{e^{\beta \varepsilon_i} - 1} \approx sV \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{e^{p^2/2mk_B T} - 1} =$$

$$\zeta_{3/2} \left(1 \right) \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} sV = N \left(\frac{T}{T_{kr}} \right)^{3/2}$$

$$\langle n_0 \rangle = N - \langle N_{wzb} \rangle = N \left[1 - \left(\frac{T}{T_{kr}} \right)^{3/2} \right]$$



$$E = s\left(\varepsilon_0 + \sum_{i=1}^{\infty} \frac{\varepsilon_i e^{-\beta\varepsilon_i}}{1-e^{-\beta\varepsilon_i}}\right) = s\int \frac{e^{-\beta p^2/2m} p^2/2m}{1-e^{p^2/2mk_B T}} V \frac{d^3p}{(2\pi\hbar^2)^3} =$$

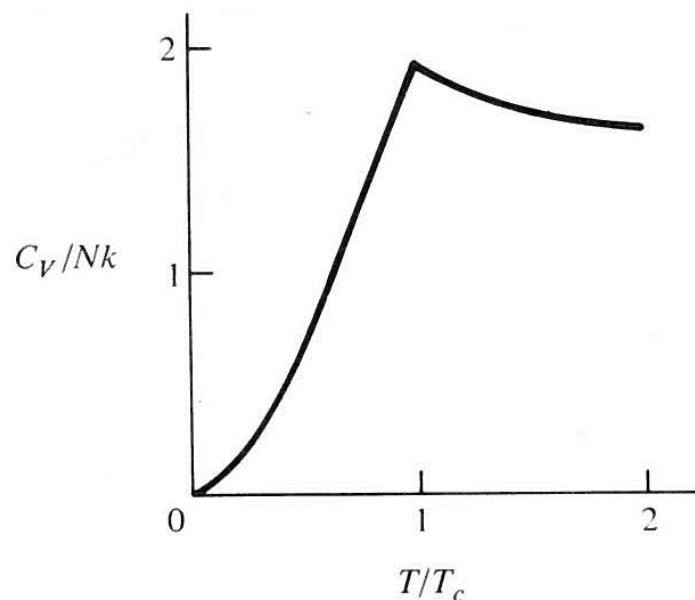
$$\frac{3}{2}k_BT\Bigg(\frac{mk_BT}{2\pi\hbar^2}\Bigg)^{3/2}\zeta_{5/2}(1)=\frac{3}{2}k_BTN_{wzb}\frac{\zeta_{5/2}(1)}{\zeta_{3/2}(1)}=\frac{3}{2}k_BT\frac{\zeta_{5/2}(1)}{\zeta_{3/2}(1)}\Bigg(\frac{T}{T_{kr}}\Bigg)^{3/2}N=$$

$$\frac{3}{2}\frac{k_BT\nu}{\Lambda^3}\zeta_{5/2}(1)\qquad\qquad\Lambda=\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{\frac{1}{2}}$$

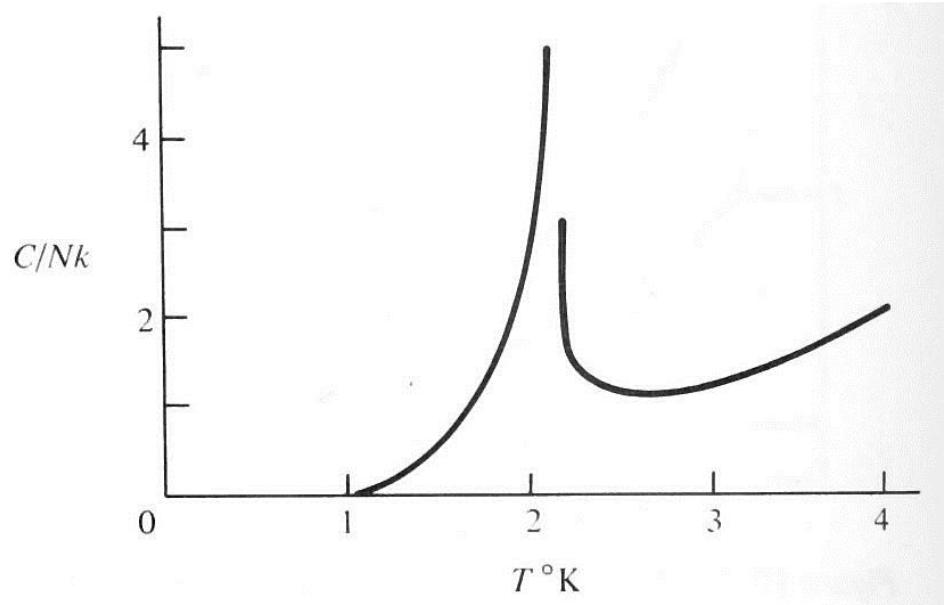
$$E/N=\begin{cases}\frac{3}{2}\frac{k_BT\nu}{\Lambda^3}\zeta_{5/2}(\lambda)&T>T_{kr}\\\frac{3}{2}\frac{k_BT\nu}{\Lambda^3}\zeta_{5/2}(1)&T<T_{kr}\end{cases}$$

$$\nu=V/N$$

$$C_v / Nk_B = \begin{cases} \frac{15}{4} \frac{\nu}{\Lambda^3} \zeta_{3/2}(\lambda) - \frac{9}{4} \frac{\zeta_{3/2}(\lambda)}{\zeta_{1/2}(\lambda)} & T > T_{kr} \\ \frac{15}{4} \frac{\nu}{\Lambda^3} \zeta_{3/2}(1) & T < T_{kr} \end{cases}$$



Układ nieoddziałyujących bozonów



Ciekły hel

Dla przejścia λ w ciekłym helu

$$C_v = \begin{cases} a + b \ln|T - T_{kr}| & T < T_{kr} \\ a' + b \ln|T - T_{kr}| & T > T_{kr} \end{cases}$$



Układ nieoddziajujących fermionów

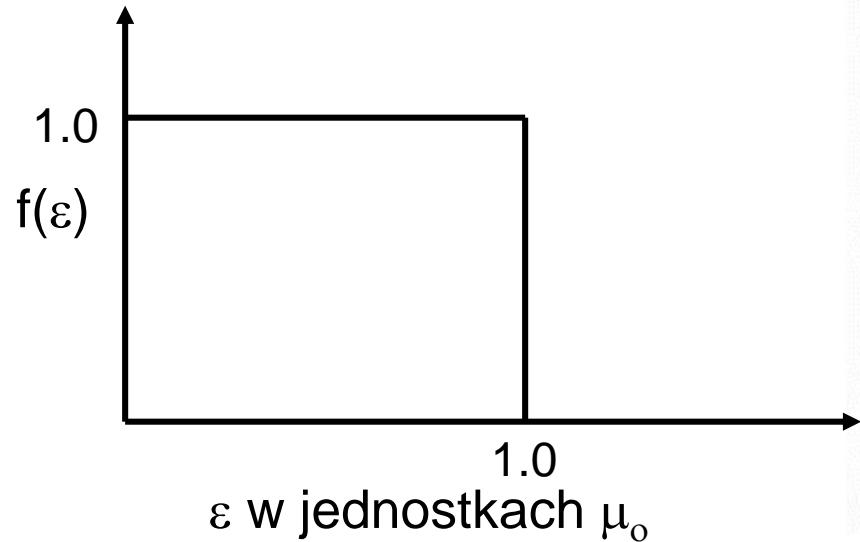
$$\Xi = \prod_i \left[1 + e^{-\beta(\varepsilon_i - \mu)} \right]$$

$$\ln \Xi = \sum_i \ln \left[1 + e^{-\beta(\varepsilon_i - \mu)} \right] \approx \int \ln \left[1 + e^{-\beta(p^2/2m - \mu)} \right] \frac{2d^3 p}{(2\pi\hbar)^3}$$

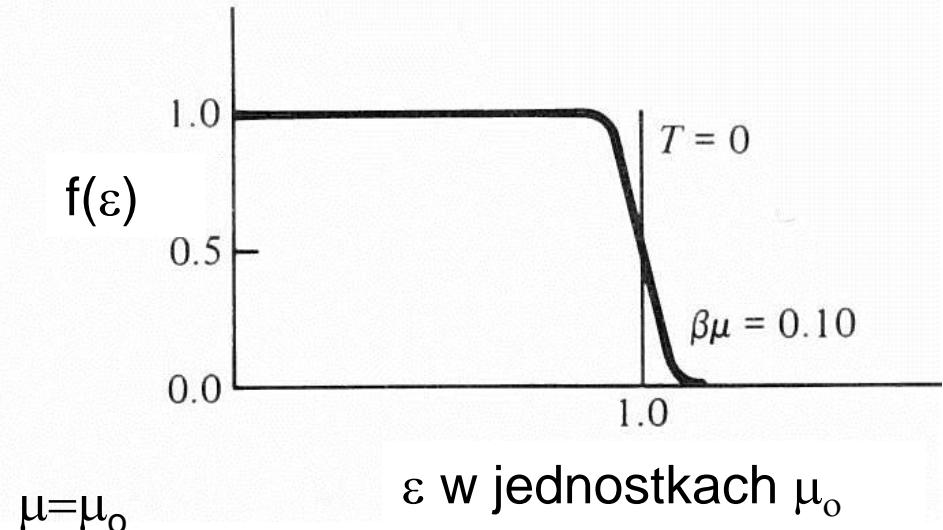
$$\rho = \frac{\langle N \rangle}{V} = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \mu} = \int \frac{e^{-\beta(p^2/2m)} e^{\beta\mu}}{1 + e^{-\beta(p^2/2m)} e^{\beta\mu}} \frac{2d^3 p}{(2\pi\hbar)^3}$$

$$\langle n_\alpha \rangle = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \varepsilon_\alpha} = \frac{e^{-\beta(\varepsilon_\alpha - \beta\mu)}}{1 - e^{-\beta(\varepsilon_\alpha - \beta\mu)}} = \frac{1}{e^{\beta(\varepsilon_\alpha - \beta\mu)} - 1}$$

$$\langle n_\alpha \rangle = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \varepsilon_\alpha} = \frac{e^{-\beta(\varepsilon_\alpha - \beta\mu)}}{1 - e^{-\beta(\varepsilon_\alpha - \beta\mu)}} = \frac{1}{e^{\beta(\varepsilon_\alpha - \beta\mu)} - 1}$$



Obsadzenie poziomu o energii ε
dla $T=0$



Obsadzenie poziomu o energii ε
dla $T>0$

μ_0 – energia Fermiego; $T_f = \mu_0/k_B T$ – temperatura Fermiego

Dla rzeczywistych układów $\mu_0 \gg k_B T$; np. dla Cu $T_f = 82000$ K.

$$u = \frac{U}{V} = \frac{1}{V} \left(-\frac{\partial \ln \Xi}{\partial \beta} + \mu N \right) = \int \frac{p^2}{2m} \frac{e^{-\beta(p^2/2m-\mu)}}{1+e^{-(p^2/2m-\mu)}} \frac{2d^3 p}{(2\pi\hbar)^3}$$

dla $T = 0$

$$u(0) = \int \left(\frac{p^2}{2m} \right) 2 \frac{4\pi p^2 dp}{(2\pi\hbar)^3} = \frac{4\pi}{(2\pi\hbar)^3} \frac{1}{2m} 2 \int_0^{p_0} p^4 dp = \frac{4\pi}{(2\pi\hbar)^3} \frac{1}{m} \frac{1}{5} p_0^5$$

$$\frac{u(0)}{\rho} = \frac{3}{5} \frac{p_0^2}{2m} = \frac{3}{5} \mu_0$$

$$\rho = \int_0^{p_0} 2 \left[\frac{4\pi p^2}{(2\pi\hbar)^3} \right] dp = 2 \left(\frac{4\pi p_0^3}{3} \right) \frac{1}{(2\pi\hbar)^3}$$

Dla $T > 0$ K

$$\rho = \frac{\langle N \rangle}{V} = \frac{1}{V\beta} \frac{\partial \ln \Xi}{\partial \mu} = \int \frac{e^{-\beta p^2/2m} e^{\beta \mu}}{1 + e^{-\beta p^2/2m} e^{\beta \mu}} \frac{2d^3 p}{(2\pi\hbar)^3}$$

$$u = \int \frac{(p^2/2m) e^{-\beta p^2/2m} e^{\beta \mu}}{1 + e^{-\beta p^2/2m} e^{\beta \mu}} \frac{2d^3 p}{(2\pi\hbar)^3}$$

$$\varepsilon = p^2/2m, \quad p = \sqrt{2m\varepsilon}, \quad dp = \sqrt{m/2\varepsilon} d\varepsilon$$

$$\rho = a \int_0^\infty \frac{\sqrt{\varepsilon} d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$u = a \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$a = \frac{4\pi(2m)^{3/2}}{(2\pi\hbar)^3}$$

$$I = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}, \quad g(\varepsilon) = \begin{cases} a\sqrt{\varepsilon} & \text{dla } \rho \\ a\varepsilon^{3/2} & \text{dla } u \end{cases}$$

$$\int_0^\infty \frac{g(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \underbrace{\int_\mu^\infty \frac{g(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}}_{x=\beta(\varepsilon-\mu)} + \int_0^\mu g(\varepsilon)d\varepsilon - \underbrace{\int_0^\mu \frac{g(\varepsilon)d\varepsilon}{e^{-\beta(\varepsilon-\mu)}+1}}_{x=-\beta(\varepsilon-\mu)}$$

$$I = \int_0^\mu g(\varepsilon)d\varepsilon + \int_0^\infty \frac{g(\mu+x/\beta)dx}{e^x+1} - \int_0^{\beta\mu} \frac{g(\mu-x/\beta)dx}{e^x+1} \frac{1}{\beta}$$

$$g(\mu \pm x/\beta) \approx g(\mu) \pm x/\beta, \quad \int_0^{\beta\mu} \cdots dx \approx \int_0^\infty \cdots dx$$

$$I \approx \int_0^\mu g(\varepsilon)d\varepsilon + \frac{2}{\beta^2} g'(\mu) \underbrace{\int_0^\infty \frac{x dx}{e^x+1}}_{= \pi^2/12} = \int_0^\mu g(\varepsilon)d\varepsilon + \frac{\pi^2 g'(\mu)}{6\beta^2}$$

$$\rho = a \int_0^{\infty} \sqrt{\varepsilon} d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \frac{a}{2\sqrt{\mu}} = \left(\frac{2a}{3} \right) \mu^{3/2} + \frac{\pi^2}{6} (k_B T)^2 \frac{a}{2\sqrt{\mu}}$$

$$\rho = (2a/3)\mu_0^{3/2}$$

$$\mu \approx \mu_0 \left(1 - \frac{\pi^2}{12} \frac{k_B^2 T^2}{\mu_0^2} \right)$$

$$u \approx a \int_0^{\mu} \varepsilon^{3/2} d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \frac{3a}{2} \sqrt{\mu} = \frac{2}{5} a \mu^{5/2} + \frac{\pi^2}{6} (k_B T)^2 \frac{3a}{2} \sqrt{\mu} \approx$$

$$\frac{2}{5} a \mu_0^{5/2} - a \mu_0^{5/2} \frac{\pi^2}{12} \frac{(k_B T)^2}{\mu_0^2} + \frac{a \pi^2}{4} (k_B T)^2 \sqrt{\mu_0} =$$

$$u_0 + \frac{a \pi^2}{6} \sqrt{\mu_0} (k_B T)^2 = u_0 + \gamma T^2, \quad \gamma = \frac{a \pi^2}{6} \sqrt{\mu_0} k_B^2$$

$$U = uV = U_0 + \gamma V T^2 = U_0 + \gamma' T^2$$

$$C_V = \frac{\partial U}{\partial T} = 2\gamma' T$$

W niskich temperaturach

$$C_V^{metal} = \underbrace{2\gamma' T}_{\text{elektryny}} + \underbrace{\alpha T^3}_{\text{drgania sieci}}$$